

BASIC ELASTICITY EQUATIONS

The majority of the literatures on the FEM treat problems in structural mechanics and hence the study and application of FEM requires a sound knowledge of elasticity equations of an elastic body. Therefore, in the following sections some of the elasticity equations, which are essential for the derivation of FEM formulations, are derived.

1. Body force and Surface force:

Consider a 3 – D body in a rectangular co-ordinate system x, y, z as shown in Fig. 1. In general the body will subjected to two types of forces namely (1) body force and (2) surface force.

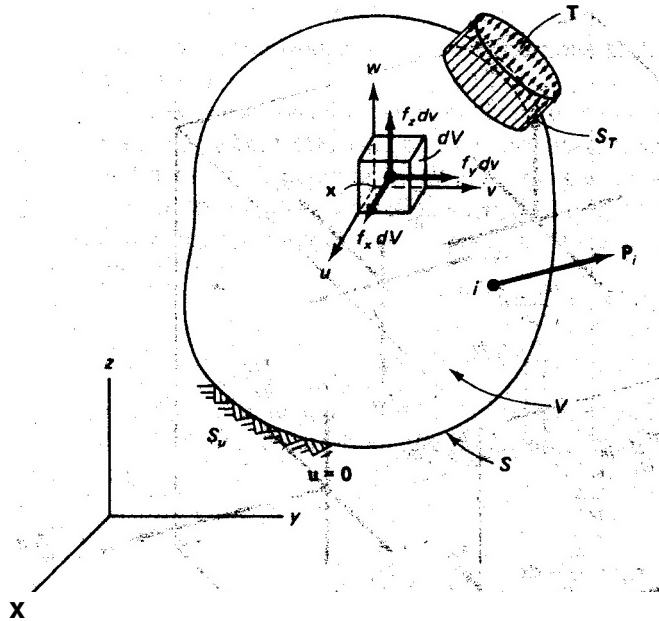


Fig. 1: 3 – D body subjected to body and surface forces

1. **Body force:** The body force acts on each volume element of the body and is expressed as force per unit volume (N/mm^3). The examples of this kind of forces are the forces due to weight of the body, centrifugal forces, gravitational force etc.

The body force per unit volume in x, y and z direction is as shown in Fig. 1 and is given by

$$\{f\} = \{f_x \quad f_y \quad f_z\}^T$$

2. **Surface force:** The surface force acts on a surface or area element of a body and is expressed as force per unit area (N/mm^2). The examples of surface forces are distributed force, contact force and action of pressure inside and/or outside the cylinder, etc. The surface force which is also called as *traction* is as shown in Fig. 1 and is given by

$$\{T\} = \{T_x \quad T_y \quad T_z\}^T$$

A load P_i acting at any point 'i' shown in Fig.1 is also surface force.

2. Rectangular stress components:

Consider a rectangular element obtained by cutting a body by three mutually perpendicular planes (x -plane, y -plane and z -plane) as shown in Fig. 2. The following normal and shear stress components act on this element.

$\sigma_x, \tau_{xy}, \tau_{xz}$ --- on x -plane

$\tau_{yx}, \sigma_y, \tau_{yz}$ --- on y -plane

$\tau_{zx}, \tau_{zy}, \sigma_z$ --- on z -plane

The plane perpendicular to x-axis is termed as x-plane and that perpendicular to y- and z- axis is termed as y and z plane respectively.

The above stress components acting on rectangular element are shown in Fig.2. Therefore the state of stress at any point within the body can be represented in the form matrix as

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Further, for the equilibrium of the element, it can be shown that

$$\tau_{xy} = \tau_{yx}; \quad \tau_{xz} = \tau_{zx} \text{ and } \tau_{yz} = \tau_{zy}$$

Therefore, the nine components of state of stress can be reduced to six components. They are

$$\{\sigma\} = \{\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}\}^T$$

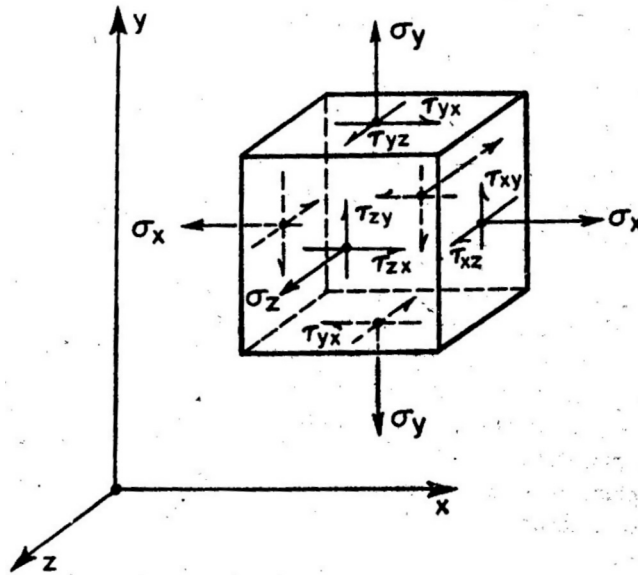


Fig. 2: stress components acting on rectangular element

Thus it is required to determine only six components of stresses for a 3 – D deformable body.

Note: In the designation of the shear stress, the first 1st subscript 'x' indicates the plane on which the stress acting and the 2nd subscript 'y' indicates the direction of the component. For example 'tau_{xy}' is the shear stress on x-plane in y-direction and 'tau_{xz}' the shear stress on x-plane in z-direction and so on.

3. Equilibrium Equation:

The deformable body is subjected to external forces and a set of internal forces developed due to deformation of the body. The external forces may be due to (i) body forces due to weight of the body, centrifugal forces, etc. that acts within the boundaries of the body and expressed as force per unit volume and (ii) surface forces which acts on the boundary surface of the body expressed as force per unit area of the surface. The equilibrium equation relates these external forces with the internal forces.

1. Equilibrium equation for a body subjected to body forces:

Consider an elemental volume of sides dx, dy and dz as shown in Fig.3. The general state of stress and their variation from point A to A' is as shown in Fig. 3. For equilibrium of elemental volume all the forces in x, y, and z directions should be individually zero.

Therefore, sum all the forces in x – direction and equate them to zero, we get

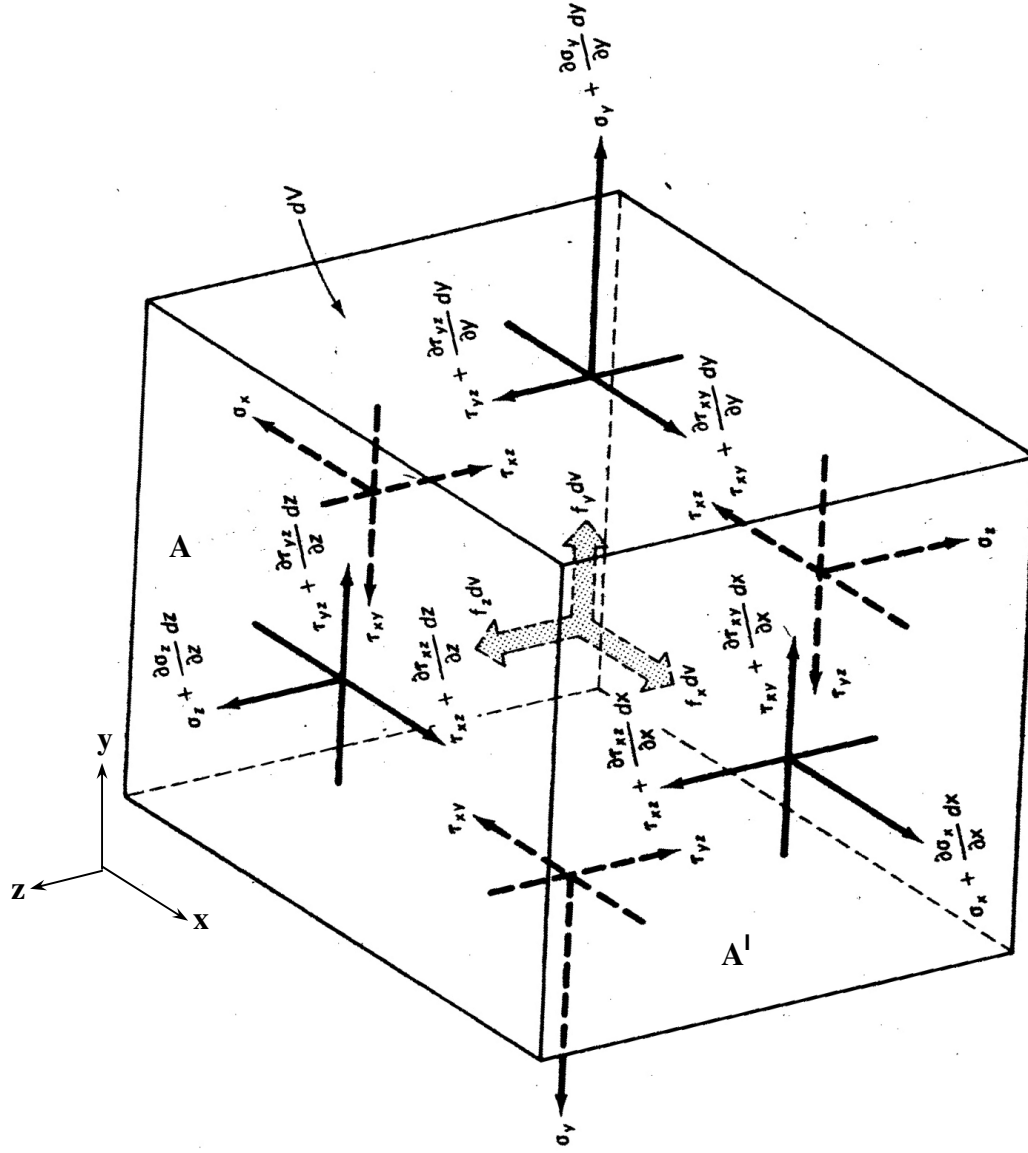


Fig. 3: Equilibrium of elemental volume

$$-\sigma_x dydz + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) dydz - \tau_{yx} dx dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx dz - \tau_{zx} dx dy + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz \right) dx dy + f_x dx dy dz = 0$$

$$-\sigma_x dydz + \sigma_x dydz + \frac{\partial \sigma_x}{\partial x} dx dy dz - \tau_{yx} dx dz + \tau_{yx} dx dz + \frac{\partial \tau_{yx}}{\partial y} dy dx dz - \tau_{zx} dx dy + \tau_{zx} dx dy + \frac{\partial \tau_{zx}}{\partial z} dz dx dy + f_x dx dy dz = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \quad \text{---(1)}$$

Following the same procedure, we can sum the forces in y and z direction and get the following equations

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0 \quad \text{---(2)}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \quad \text{---(3)}$$

In the above equations f_x, f_y and f_z are the body force components along x, y and z directions.

Three more remaining conditions for equilibrium are the moments about the axis x, y and z are individually zero. Therefore, taking moments about the axis x, y and z it can be shown that

$$\tau_{xy} = \tau_{yx}; \quad \tau_{xz} = \tau_{zx} \text{ and } \tau_{yz} = \tau_{zy}$$

2. Equilibrium equation for a body subjected to surface forces:

Consider an elemental tetrahedron ABCD on the surface (boundary) of a solid as shown in Fig.4. Let T_x, T_y and T_z be the components of surface forces (force per unit area) acting on the inclined surface ABC.

For the equilibrium of Tetrahedron, the sum of all forces in x, y and z directions must be individually zero.

Therefore, sum all the forces in x-direction and equate them to zero, we get

$$-\sigma_x \text{ Area BCD} - \tau_{yx} \text{ Area ACD} - \tau_{zx} \text{ Area ABD} + T_x \text{ Area ABC} = 0 \quad \text{---(a)}$$

$$\text{But, Area BCD} = \text{Projected area of ABC on x-plane} = dA * l \quad \text{---(i)}$$

$$\text{Area ACD} = \text{Projected area of ABC on y-plane} = dA * m \quad \text{---(ii)}$$

$$\text{Area ABD} = \text{Projected area of ABC on z-plane} = dA * n \quad \text{---(iii)}$$

Where, dA is the area of inclined plane ABC and l, m, n are direction cosines and are given by

$$l = \cos \theta_x; \quad m = \cos \theta_y \text{ and } n = \cos \theta_z.$$

Where, $\theta_x, \theta_y, \theta_z$ are the angles of inclination of plane ABC with x, y and z planes respectively.

Substituting Eqns (i)-(iii) into Eqn. (a) and rearranging we get

$$\sigma_x l + \tau_{yx} m + \tau_{zx} n = T_x \quad \text{---(1)}$$

Similarly, for the equilibrium of tetrahedron in y and z directions, we can write

$$\tau_{xy} l + \sigma_y m + \tau_{zy} n = T_y \quad \text{---(2)}$$

$$\tau_{xz} l + \tau_{yz} m + \sigma_z n = T_z \quad \text{---(3)}$$

The above Eqns. (1) to (3) are also termed as 'boundary conditions' and they must satisfy on the boundary of the body.

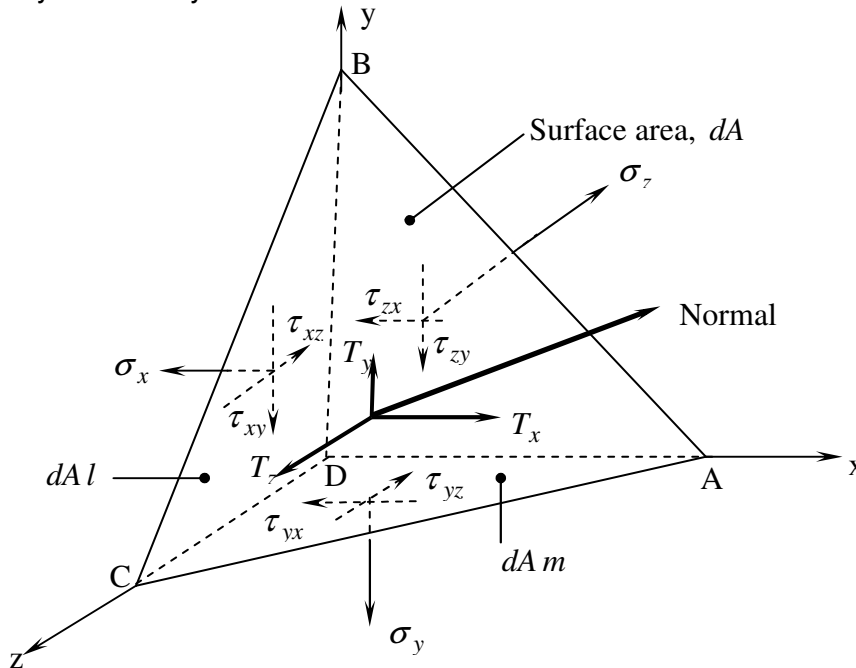


Fig. 4: Elemental Tetrahedron

4. Stress – Strain Relation or Constitutive Relation:

The stress – strain relation for an isotropic linear elastic material can be obtained from the generalized Hooke's law. For an isotropic material, the two material properties namely Young's modulus ' E ' and Poisson's ratio ' μ ' are required to describe the stress – strain relation.

Consider a elemental cube inside a body. The strains in x, y and z directions can be expressed as

$$\epsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} \quad \text{---(1)}$$

$$\epsilon_y = -\mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} \quad \text{---(2)}$$

$$\epsilon_z = -\mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} + \frac{\sigma_z}{E} \quad \text{---(3)}$$

and shear strains as

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\mu)}{E} \tau_{xy} \quad \text{---(4)} \quad \because G = \frac{E}{2(1+\mu)}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{2(1+\mu)}{E} \tau_{yz} \quad \text{---(5)}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{2(1+\mu)}{E} \tau_{zx} \quad \text{---(6)}$$

The above equations can also be written in matrix form as

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

In short form

$$\{\epsilon\} = [C]\{\sigma\} \quad \text{---(A)}$$

The above equation relates stress with strain i.e. if six components of stresses are known then six components of strains can be determined from the above equation. It can also be possible to relate the strain with stress by solving Eqns. (1) to (6) and is expressed as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\mu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\mu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\mu)}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

In short form

$$\{\sigma\} = [D]\{\epsilon\} \quad \text{---(B)}$$

where, matrices **C** and **D** in Eqns. (A) and (B) are called as *constitutive or material or elasticity matrix*. The matrix **C** in Eqn. (A) relates stress to strain and matrix **D** in Eqn. (B) relates strain to stress and these matrices are square, symmetric matrices.

5. Strain – Displacement Relations:

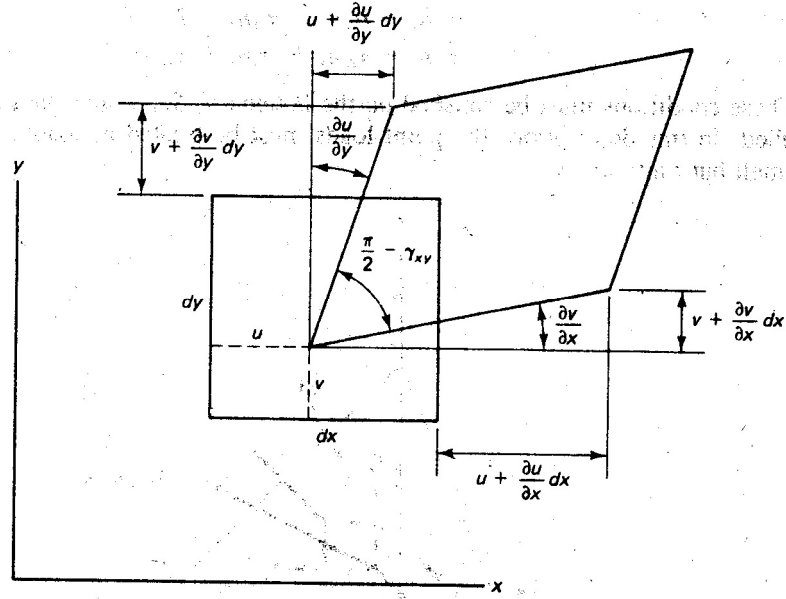


Fig. 5: Deformed elemental surface

Consider a plane ABCD before loading as shown in Fig. 5. As a result of loading, the plane occupies the position and shape as shown.

By definition, the normal strain is the ratio of change in length to original length. Therefore, the strain in x – direction is given by

$$\epsilon_x = \frac{\text{Change in length}}{\text{Original length}} = \frac{u + \frac{\partial u}{\partial x} dx - u}{dx} = \frac{\partial u}{\partial x} \quad \text{---(1)}$$

Similarly,

$$\epsilon_y = \frac{\partial v}{\partial y} \quad \text{---(2)}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \quad \text{---(3)}$$

The shear strain is defined as the amount of change in right angle of plane. Therefore,

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \text{---(4)}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \text{---(5)}$$

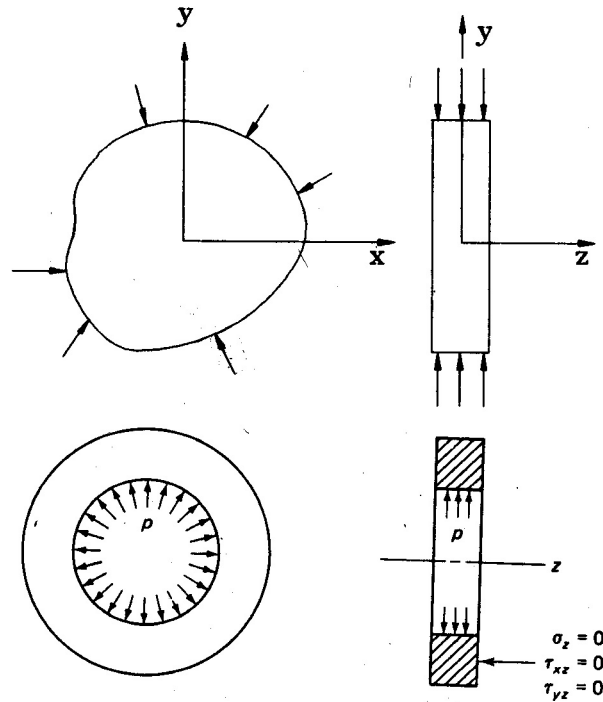
$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad \text{---(6)}$$

6. Two – Dimensional Problems:

The 3 – D problems in solid mechanics can be analyzed using FEM. In such cases computational time increases due to more number of equations and hence analysis becomes costlier. However, there are many situations where geometry and loading configurations permit to reduce these 3 – D problems into 2 – D problems and accurate results can be obtained. These 2 – D problems are formulated as

(i) plane stress (ii) plane strain and (iii) axi – symmetric problems.

1. Plane Stress Problems:



(a) Thin plate with in – plane loading

(b) Ring press fitted on a shaft

Fig. 6: Plane stress problems

The problems involving a very small dimension in z – direction are referred as plane stress problems. The examples of plane stress problems are a thin plate with in – plane loading (Fig. 6(a)) and a ring press fitted on a shaft (Fig. 6(b)). In this problem there is no load applied on the surface of the plate i.e. in z – direction. Hence, the stress components in z – direction is zero.

i.e. $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

The stress – strain relation for this plane stress problem reduces to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\text{or } \{\sigma\} = [D]\{\epsilon\}$$

Also strain in z – direction is

$$\epsilon_z = -\frac{\mu}{1-\mu}(\epsilon_x + \epsilon_y)$$

2. Plane Strain problem:

Problems involving a long body whose geometry and loading do not vary significantly in the longitudinal direction (i.e. in z – direction) are referred as plain strain problems. Examples of plane strain problems are long strip footing, long cylinder and earth dam as shown in Figs. 7 (a) to 7(c).

In these problems, the displacement component in z – direction is zero at every cross section and it is assumed that the displacements are the function of x and y coordinates only.

For plane strain problems, $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$ and the stress strain relation reduces to

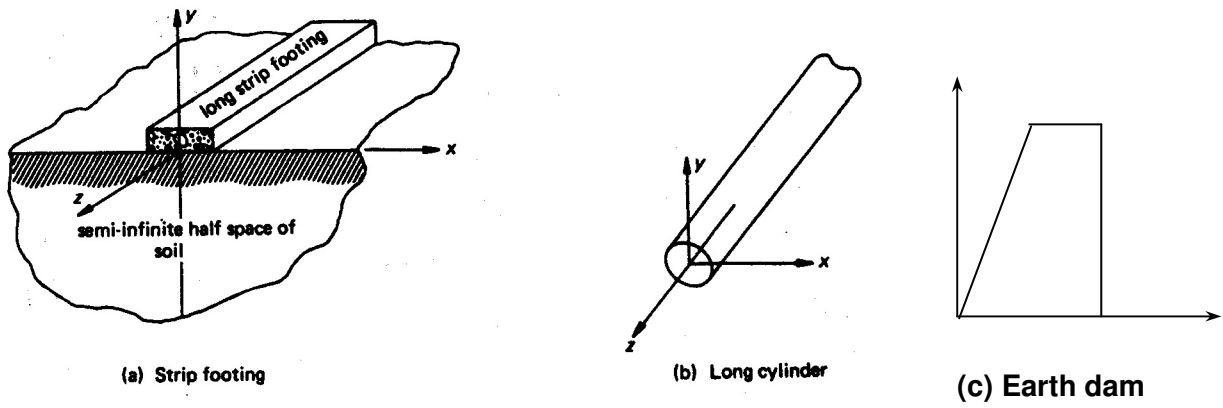


Fig. 7: Plane strain problems

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

or $\{\sigma\} = [D]\{\epsilon\}$

Also stress in z – direction is given by

$$\sigma_z = \mu(\sigma_x + \sigma_y)$$

3. Axi – Symmetric Problems:

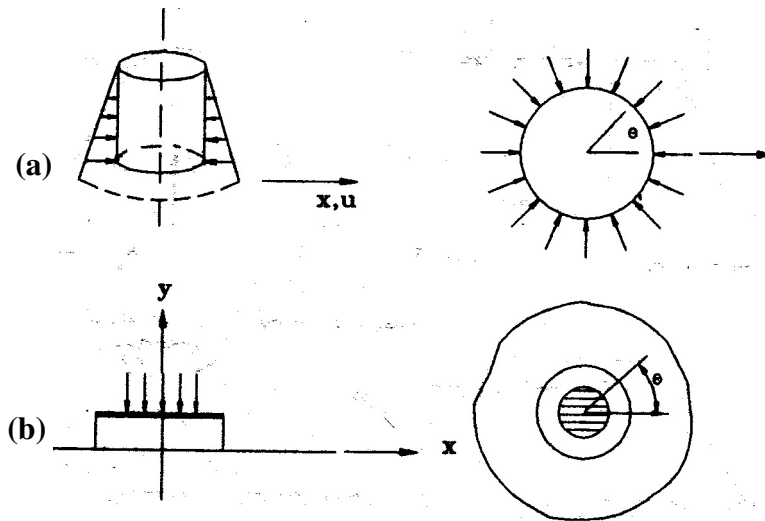


Fig. 8: Axi-Symmetric problems

The problems involve solid of revolution (axi-symmetric solids) and subjected to axially symmetric loading are referred as axi – symmetric problems. Examples of this problems are a circular cylinder loaded by uniform internal or external pressure (Fig. 8(a)), a circular footing resting on a soil mass (Fig. 8(b)), pressure vessels, rotating wheels, flywheels, etc. The deformation being symmetrical w.r.t y – axis and stress components are independent of the angle ' θ ' and all derivatives w.r.t ' θ ' vanish.

Therefore, the components $w, \gamma_{x\theta}, \gamma_{y\theta}, \tau_{x\theta}, \tau_{y\theta} = 0$. The stress strain relation for this problem reduces to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} (1-\mu) & \mu & \mu & 0 \\ \mu & (1-\mu) & \mu & 0 \\ \mu & \mu & (1-\mu) & 0 \\ 0 & 0 & 0 & \frac{(1-2\mu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_\theta \\ \gamma_{xy} \end{Bmatrix}$$

$$\text{or } \{\sigma\} = [D]\{\varepsilon\}$$

Prob: Derive the stress strain relations for plane stress and plane strain problems.

Plane Stress Problem: consider a thin plate subjected to in-plane loading on its edge surface as shown in Fig. 6(a). For a plane stress problem, $\sigma_z = \tau_{xz} = \tau_{yz} = 0.0$. Then strain in x and y – directions is given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} \quad \text{---(1)}$$

$$\varepsilon_y = -\mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} \quad \text{---(2)}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\mu)}{E} \tau_{xy} \quad \text{---(4)} \quad \because G = \frac{E}{2(1+\mu)}$$

Add Eqns. (1) and (2) we get

$$\varepsilon_x + \varepsilon_y = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} = (1-\mu) \frac{\sigma_x}{E} + (1-\mu) \frac{\sigma_y}{E}$$

$$\therefore (1-\mu) \frac{\sigma_y}{E} = \varepsilon_x + \varepsilon_y - (1-\mu) \frac{\sigma_x}{E}$$

$$\text{Or } \frac{\sigma_y}{E} = \frac{\varepsilon_x}{(1-\mu)} + \frac{\varepsilon_y}{(1-\mu)} - \frac{\sigma_x}{E}$$

Substituting into Eqn. (1) we get

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu \left(\frac{\varepsilon_x}{(1-\mu)} + \frac{\varepsilon_y}{(1-\mu)} - \frac{\sigma_x}{E} \right) = \frac{\sigma_x}{E} + \mu \frac{\sigma_x}{E} - \frac{\mu}{(1-\mu)} \varepsilon_x - \frac{\mu}{(1-\mu)} \varepsilon_y$$

$$\varepsilon_x = (1+\mu) \frac{\sigma_x}{E} - \frac{\mu}{(1-\mu)} \varepsilon_x - \frac{\mu}{(1-\mu)} \varepsilon_y$$

$$(1+\mu) \frac{\sigma_x}{E} = \varepsilon_x + \frac{\mu}{(1-\mu)} \varepsilon_x + \frac{\mu}{(1-\mu)} \varepsilon_y$$

$$(1+\mu) \frac{\sigma_x}{E} = \frac{\varepsilon_x(1-\mu) + \mu \varepsilon_x}{(1-\mu)} + \frac{\mu}{(1-\mu)} \varepsilon_y = \frac{\varepsilon_x - \mu \varepsilon_x + \mu \varepsilon_x}{(1-\mu)} + \frac{\mu}{(1-\mu)} \varepsilon_y = \frac{\varepsilon_x}{(1-\mu)} + \mu \frac{\varepsilon_y}{(1-\mu)}$$

$$\sigma_x = \frac{E \varepsilon_x}{(1+\mu)(1-\mu)} + \mu \frac{E \varepsilon_y}{(1+\mu)(1-\mu)} = \frac{E}{(1-\mu^2)} \varepsilon_x + \frac{E}{(1-\mu^2)} \mu \varepsilon_y \quad \text{---(i)}$$

Similarly, we can write

$$\sigma_y = \frac{E}{(1-\mu^2)} \mu \varepsilon_x + \frac{E}{(1-\mu^2)} \varepsilon_y \quad \text{---(ii)}$$

$$\tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy} \quad (\text{From Eqn. (3)})$$

$$\tau_{xy} = \frac{E}{2(1+\mu)} \times \frac{(1-\mu)}{(1-\mu)} \gamma_{xy} = \frac{E}{(1-\mu^2)} \frac{(1-\mu)}{2} \gamma_{xy} \quad \text{---(iii)}$$

Eqns. (i) – (iii) can be written in matrix form as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\mu}{2}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\text{or } \{\sigma\} = [D]\{\varepsilon\}$$

Also strain in z – direction is $\varepsilon_z = -\frac{\mu}{E}(\sigma_x + \sigma_y)$

Plane Strain Problem: For a plane strain problem, the strain in one of the principle direction (usually in z – direction) is negligible. Therefore, $\varepsilon_x = \gamma_{xz} = \gamma_{yz} = 0.0$. The strain components are given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} \quad \text{---(1)}$$

$$\varepsilon_y = -\mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} \quad \text{---(2)}$$

$$\varepsilon_z = 0 = -\mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} + \frac{\sigma_z}{E} \quad \text{---(3)}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\mu)}{E} \tau_{xy} \quad \text{---(4)} \quad \because G = \frac{E}{2(1+\mu)}$$

Add Eqns. (1), (2) & (3) we get

$$\begin{aligned} \varepsilon_x + \varepsilon_y + 0 &= \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} + \frac{\sigma_z}{E} \\ \varepsilon_x + \varepsilon_y &= \frac{(1-2\mu)}{E} \sigma_x + \frac{(1-2\mu)}{E} (\sigma_y + \sigma_z) \\ \therefore \frac{(\sigma_y + \sigma_z)}{E} &= \frac{\varepsilon_x}{(1-2\mu)} + \frac{\varepsilon_y}{(1-2\mu)} - \frac{\sigma_x}{E} \quad \text{---(5)} \end{aligned}$$

Eqn. (1) can be re-written as

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu \left(\frac{\sigma_y + \sigma_z}{E} \right)$$

After substitution for $\left(\frac{\sigma_y + \sigma_z}{E} \right)$ from Eqn. (5) and simplification, we get

$$\sigma_x = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \varepsilon_x + \frac{E\mu}{(1+\mu)(1-2\mu)} \varepsilon_y \quad \text{---(i)}$$

Similarly we can write

$$\sigma_y = \frac{E\mu}{(1+\mu)(1-2\mu)} \varepsilon_x + \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \varepsilon_y \quad \text{---(ii)}$$

Eqn. (4) can be written as

$$\tau_{xy} = \frac{E}{2(1+\mu)} \gamma_{xy} = \frac{E}{(1+\mu)(1-2\mu)} \times \frac{(1-2\mu)}{2} \gamma_{xy} \quad \text{---(iii)}$$

Eqns. (i) – (iii) can be written in matrix form as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\text{or } \{\sigma\} = [D]\{\epsilon\}$$

Also stress in z – direction is derived as

$$\epsilon_z = 0 = -\mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} + \frac{\sigma_z}{E} = -\mu(\sigma_x + \sigma_y) + \sigma_z$$

$$\therefore \sigma_z = \mu(\sigma_x + \sigma_y)$$

7. Basic Principle of solid Mechanics:

The principle of *virtual work* and *principle of minimum potential energy* methods are more powerful and widely used approaches in the formulation of element equations and its structural properties.

1. Principle of virtual work:

A force ' P ' is said to do work if it give rise to a displacement ' r '. Then work ($W = p*r$) is the product of force and the corresponding displacement. When the force ' P ' and displacement ' r ' are unrelated by '*cause – effect relation*', the work $W = P*r$ is called *virtual work*. Hence the virtual may be caused by *true forces* moving through *imaginary displacement* or *vice – versa*. Therefore, the principle of virtual work can be divided into two parts namely: (1) The principle of virtual displacement and (2) The principle of virtual forces.

The principle of virtual displacement is based on the virtual work done by the true forces moving through the virtual displacement. This principle establishes the condition of equilibrium and is used in the displacement models of the FEM. On the other hand, the principle of virtual forces based on the virtual work done by the virtual forces in moving through the true displacements. This principle establishes the compatibility conditions and it is used in equilibrium models of FEM.

Principle of virtual displacement:

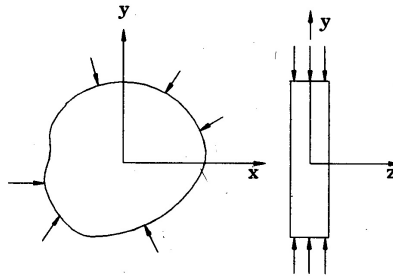


Fig. 9: Two – dimensional body

Consider a 2 – D body as shown in Fig. 9. The equilibrium conditions to be satisfied by the stresses within the body and on the boundary (surface) are as follows:

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y &= 0 \end{aligned} \right\} \quad \text{---(a)}$$

and

$$\left. \begin{aligned} \sigma_x l + \tau_{xy} m &= T_x \\ \tau_{xy} l + \sigma_y m &= T_y \end{aligned} \right\} \quad \text{---(b)}$$

Multiplying the equilibrium equation (a) by virtual displacements δu and δv respectively and integrating over area of the solid, we get

$$\iint_A \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x \right) \delta u + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y \right) \delta v \right] dx dy = 0$$

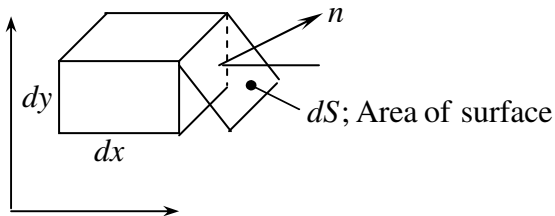
$$\underbrace{\iint_A \left(\frac{\partial \sigma_x}{\partial x} \delta u + \frac{\partial \sigma_y}{\partial y} \delta v \right) dx dy}_{\text{Part-1}} + \underbrace{\iint_A \left(\frac{\partial \tau_{xy}}{\partial y} \delta u + \frac{\partial \tau_{xy}}{\partial x} \delta v \right) dx dy}_{\text{Part-2}} + \underbrace{\iint_A (f_x \delta u + f_y \delta v) dx dy}_{\text{Part-3}} = 0 \text{ --- (c)}$$

Part wise integration of individual terms of the above Eqn. (c) is as follows:

Part-1:

$$\begin{aligned} \iint_A \left(\frac{\partial \sigma_x}{\partial x} \delta u + \frac{\partial \sigma_y}{\partial y} \delta v \right) dx dy &= \int_S \sigma_x \delta u dy + \sigma_y \delta v dx - \iint_A \left(\sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} \right) dx dy \\ \Rightarrow \int_S \sigma_x l \delta u dS + \sigma_y m \delta v dS &- \iint_A \left(\sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} \right) dx dy \end{aligned}$$

Note:


$$dy * 1 = \text{Projection of dS on x - plane}$$

$$dy = dS * Cos\theta_x = ldS$$

Similarly, $\begin{matrix} dx = mdS \\ dz = ndS \end{matrix}$

Part-2:

$$\begin{aligned} \iint_A \left(\frac{\partial \tau_{xy}}{\partial y} \delta u + \frac{\partial \tau_{xy}}{\partial x} \delta v \right) dx dy &= \int_S \tau_{xy} m \delta u dS + \tau_{xy} l \delta v dS - \iint_A \left(\tau_{xy} \frac{\partial \delta u}{\partial y} + \tau_{xy} \frac{\partial \delta v}{\partial x} \right) dx dy \\ \Rightarrow \int_S (\tau_{xy} m \delta u + \tau_{xy} l \delta v) dS &- \iint_A \left(\tau_{xy} \frac{\partial \delta u}{\partial y} + \tau_{xy} \frac{\partial \delta v}{\partial x} \right) dx dy \end{aligned}$$

Part-3: It remain as it is.

Substitute into Eqn. (c) and rearranging we get,

$$\begin{aligned}
& - \iint_A \left[\sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} + \tau_{xy} \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) \right] dx dy + \int_S [(\sigma_x l + \tau_{xy} m) \delta u + (\tau_{xy} l + \sigma_x m) \delta v] dS \\
& + \iint_A (f_x \delta u + f_y \delta v) dx dy = 0
\end{aligned}$$

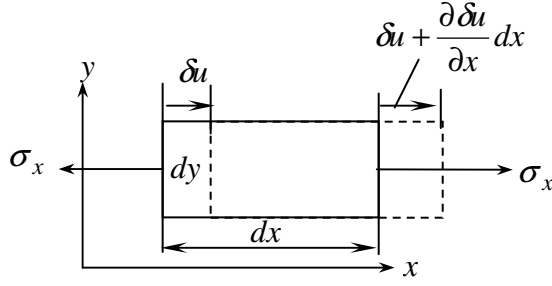
Substitute from Eqn. (b) into 2nd part of the above equation, we get:

$$-\iint_A \left[\sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} + \tau_{xy} \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) \right] dx dy + \int_S [T_x \delta u + T_y \delta v] dS + \iint_A (f_x \delta u + f_y \delta v) dx dy = 0$$

---(d)

(a) *Virtual work done by internal forces:*

In the following, it can be shown that the first part of Eqn. (d) represents the virtual work done by the internal forces.



Consider a element $dx dy$ of unit thickness subjected to stress σ_x only as shown in Fig. Let a virtual displacement ' δu ' be given to the element, then the internal virtual work done by the stress σ_x is given by

$$\text{Internal virtual work} = \sigma_x (dy * 1) \left[\delta u + \frac{\partial \delta u}{\partial x} dx \right] - \sigma_x (dy * 1) \delta u = \sigma_x \frac{\partial \delta u}{\partial x} dx dy$$

If we extend the above concept for the stresses σ_y & τ_{xy} also, we can say that the 1st term in L.H.S of Eqn. (d) is the virtual work done by the internal forces.

$$\text{Therefore, } \delta U = \iint_A \left[\sigma_x \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} + \tau_{xy} \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) \right] dx dy$$

Further, from the above Fig. the virtual strain is given by

$$\delta \epsilon_x = \frac{\delta u + \frac{\partial \delta u}{\partial x} dx - \delta u}{dx} = \frac{\partial \delta u}{\partial x}$$

Extending this for 2 – D problem and substitute into the above equation we get,

$$\delta U = \iint_A [\sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \tau_{xy} \delta \gamma_{xy}] dx dy$$

If we extend the concept for 3 – D problem, the above equation becomes as

$$\delta U = \iint_A [\sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \sigma_z \delta \epsilon_z + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{zx} \delta \gamma_{zx}] dx dy$$

$$\text{In matrix form, } \delta U = \iiint_V \{\delta \epsilon\}^T \{\sigma\} dV \quad \text{---(e)}$$

$$\text{Where, } \{\delta \epsilon\}^T = \{\delta \epsilon_x \quad \delta \epsilon_y \quad \delta \epsilon_z \quad \delta \gamma_{xy} \quad \delta \gamma_{yz} \quad \delta \gamma_{zx}\}$$

$$\{\sigma\} = \{\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}\}^T$$

Eqn. (e) is the virtual work done by internal forces.

(b) Virtual work done by external forces:

The 2nd and 3rd terms in L.H.S of Eqn. (d) represents the virtual work done by the external forces.

Therefore,

$$\delta W_e = \iint_A (f_x \delta u + f_y \delta v) dx dy + \int_S [T_x \delta u + T_y \delta v] dS$$

For 3 – D problem, the above equation can be written as

$$\delta W_e = \iiint_V (f_x \delta u + f_y \delta v + f_z \delta w) dx dy dz + \iint_S [T_x \delta u + T_y \delta v + T_z \delta w] dS$$

In matrix form,

$$\delta W_e = \iiint_V \{\delta u\}^T \{f\} dV + \iint_S \{\delta u\}^T \{T\} dS \quad \text{---(f)}$$

$$\text{where, } \{\delta u\}^T = \{\delta u \quad \delta v \quad \delta w\}; \quad \{f\} = \{f_x \quad f_y \quad f_z\}^T \text{ and } \{T\} = \{T_x \quad T_y \quad T_z\}^T$$

Eqn. (f) is the virtual work done by external forces. Thus, from Eqns. (d), (e) and (f) we can write

$$-\delta U + \delta W_e = 0 \quad \text{or} \quad \delta U = \delta W_e \quad \text{---(g)}$$

(c) *Principle of virtual work*: From Eqn. (g), the principle of virtual work can be stated as “for equilibrium to exist the total internal virtual work done is equal to the total external virtual work done”.

2. Principle of Minimum Potential energy:

The principle of minimum potential energy states that “for conservative system of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable”. According to the principle of minimum potential energy for the system to be stable, the derivative of total potential energy w.r.t displacement ‘ u ’ should be zero.

Note: Kinematically admissible displacement fields are those, which satisfy compatibility and the boundary conditions.

For a structure or deformable body, the total potential energy is the sum of the strain energy due to internal forces (i.e. due to stresses) and potential energy due to external forces (i.e. due to loads). Therefore,

$$\pi = U + W_p$$

where, U = strain energy and W_p = potential energy due to external forces or work potential.

(a) *Strain energy (U)*:

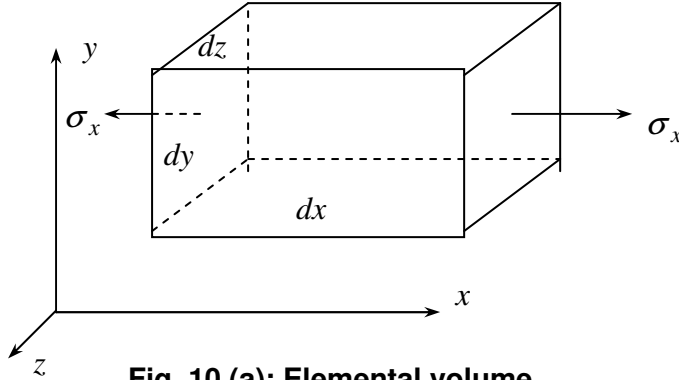


Fig. 10(a): Elemental volume

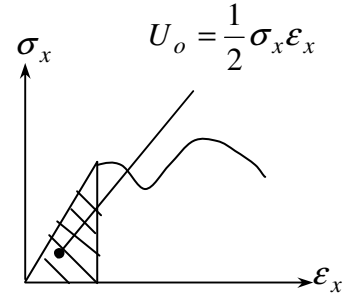


Fig. 10(b): stress vs strain

Consider an elemental volume dx , dy and dz subjected to stress ‘ σ_x ’ only as shown in Fig. 10(a). Let ‘ ϵ_x ’ be the strain due to this stress. Then the stress strain diagram is as shown in fig. 10(b).

Let U_o be the strain energy density (i.e. strain energy per unit volume). Then,

$$U_o = \frac{1}{2} \sigma_x \epsilon_x \quad \text{i.e. area of shaded portion in stress strain diagram.}$$

Extending the above expression for a 3 – D stress system, we get

$$U_o = \frac{1}{2} [\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}]$$

Or in matrix form,

$$U_o = \frac{1}{2} \{\sigma\}^T \{\epsilon\} \quad \text{---(a)}$$

where, $\{\sigma\}^T = [\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}]$ and $\{\epsilon\} = \{\epsilon_x \quad \epsilon_y \quad \epsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}\}$

The Eqn. (a) represents the strain energy per unit volume. The strain energy for the entire body is obtained by integrating the above equation over volume.

$$U = \iiint_V U_o dV = \frac{1}{2} \iiint_V \{\sigma\}^T \{\epsilon\} dV \quad \text{---(b)}$$

(b) Potential energy due to external forces or work potential (W_p):

The potential energy due to external forces (i.e. due to body & surface forces) is given by

$$W_p = - \iiint_V [f_x u + f_y v + f_z w] dV - \iint_S [T_x u + T_y v + T_z w] dS$$

where, u , v & w are the displacement components in x , y & z directions respectively. The above equation can be written in matrix form as

$$W_p = - \iiint_V \{u\}^T \{f\} dV - \iint_S \{u\}^T \{T\} dS$$

If the body is also subjected to concentrated load at point ' i ', then the above equation becomes as

$$W_p = - \iiint_V \{u\}^T \{f\} dV - \iint_S \{u\}^T \{T\} dS - \sum \{u_i\}^T \{P_i\} \quad \text{---(c)}$$

where, $\{u_i\}^T = \{u_i \quad v_i \quad w_i\}$ is the displacement components corresponding to point ' i ' where concentrated load ' P_i ' is acting.

(c) Total potential energy (π):

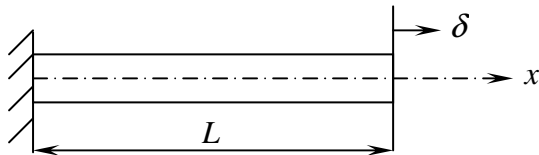
The total potential energy is given by $\pi = U + W_p$

Substituting from Eqns. (b) and (c) we get

$$\pi = \frac{1}{2} \iiint_V \{\sigma\}^T \{\epsilon\} dV - \iiint_V \{u\}^T \{f\} dV - \iint_S \{u\}^T \{T\} dS - \sum \{u_i\}^T \{P_i\}$$

Problems:

Prob. 1: For the rod shown in figure strain at any point x is given by $\epsilon_x = 1 + 2x^2$. Find the tip displacement δ . (VTU, Jan 2008)



Solutions: Strain of the rod in x – direction is given by

$$\epsilon_x = \frac{\partial u}{\partial x} = 1 + 2x^2$$

Integrate the above equation once; we get the deformation at any point along the rod.

$$u = x + 2 \frac{x^3}{3} + C \quad \text{---(a)}$$

where C is integration constant that can be determined using the following boundary condition: i.e. at $x = 0, u = 0$. Substitute into Eqn. (a), we get

$$\therefore 0 = 0 + \frac{2}{3} * 0 + C \text{ or } C = 0$$

$$\therefore u = x + \frac{2}{3} x^3 \quad \text{---(b)}$$

Eqn. (b) represents the displacement field at any point of rod along its length. Hence, tip displacement δ is obtained by substituting $x = L$ (i.e. $u|_{x=L} = \delta$) into Eqn.(b).

$\therefore \delta = L + \frac{2}{3}L^3$ is the tip displacement.

Prob. 2: In a plane strain problem, $\sigma_x = 20\text{Mpa}$, $\sigma_y = -10\text{Mpa}$, Young's modulus = 200 Gpa and poisson's ratio = 0.3. Determine the value of stress σ_z .

Solution: The strain of a 3D component in z-direction is expressed as

$$\epsilon_z = -\mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

For a plain strain problem, the strain in z-direction is zero (i.e. $\epsilon_z = 0$). Substitute into above equation and modify, we get

$$\frac{\sigma_z}{E} = \frac{\mu}{E}(\sigma_x + \sigma_y) \text{ Or } \sigma_z = \mu(\sigma_x + \sigma_y) = 0.3(20 - 10) = 3\text{Mpa}$$

Prob.3: Explain the theorem of minimum potential energy and derive an expression for total potential energy for a 1 – dimensional bar subjected to an axial force.

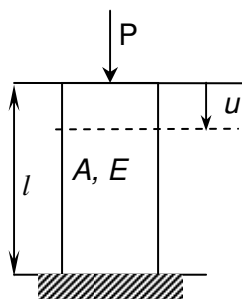
(VTU, July 2002)

Solutions:

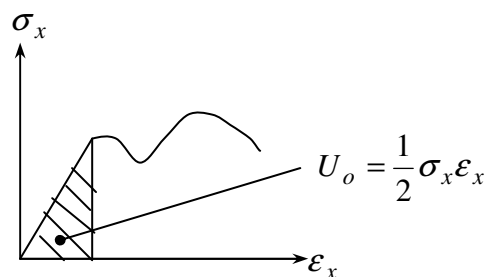
(a) Define principle of potential energy.

(b) To derive an expression for total potential energy (π):

Consider a 1 – D bar subjected to an external load P as shown in Fig. (a). The stress strain diagram is as shown in Fig. (b).



(a)



(b)

Let A = C/s area of the bar
 E = Young's modulus
 u = Deformation of bar at free end
 σ_x = Stress due to load P
 ϵ_x = Strain due to stress

(i) Strain energy (U):

Let U_o be the strain energy density of a bar. Then $U_o = \frac{1}{2} \sigma_x \epsilon_x$, area of shaded portion in stress strain diagram.

The total strain energy of entire bar is obtained by integrating the above equation over volume of bar.

$$U = \frac{1}{2} \int_V \sigma_x \epsilon_x dV \quad \text{---(a)}$$

But for a bar $\sigma_x = E\varepsilon_x$ and strain $\varepsilon_x = \frac{\text{Change in length}}{\text{Original length}} = \frac{(l-u)-l}{l} = -\frac{u}{l} = \frac{u}{l}$

Disregard – ve sign that representing compressive strain.

$\therefore \sigma_x = E \frac{u}{l}$. Substitute for σ_x and ε_x into Eqn. (a) we get

$$U = \frac{1}{2} \int_0^l E \frac{u^2}{l^2} A dx \quad \text{for a bar } dV = A dx$$

$$U = AE \frac{u^2}{2l^2} x \Big|_0^l = AE \frac{u^2}{2l^2} * l = \frac{1}{2} \frac{AE}{l} u^2$$

$$U = \frac{1}{2} Ku^2 \quad \text{---(b)}$$

where, $K = \frac{AE}{l}$, stiffness of bar

(ii) Work potential (W_p):

As there is no body and surface forces acting on the bar, the work potential is only due to axial force and is give by

$$W_p = -Pu \quad \text{---(c)}$$

(iii) Total potential energy (π):

The total P.E is $\pi = U + W_p$. Substitute for U and W_p we get

$$\pi = \frac{1}{2} Ku^2 - Pu \text{ is the expression for total potential energy for 1 – D bar.}$$

Prob.4: A rectangular bar is subjected to an axial load P as shown in Fig. Derive an expression for potential energy and hence determine the extreme value of the potential energy for the following data.
(VTU, Feb.2002)

Modulus of elasticity, $E = 200 \text{ Gpa}$

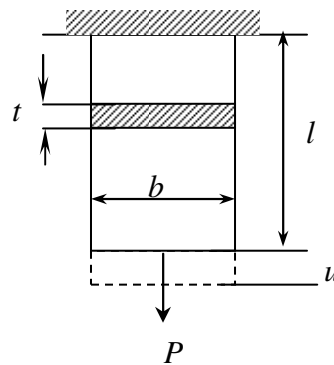
Load, $P = 3 \text{ KN}$

Length of bar, $l = 100 \text{ mm}$

Width of bar, $b = 20 \text{ mm}$

Thickness of bar, $t = 10 \text{ mm}$

Also state its equilibrium stability.



Solution:

(a) To derive an expression for total P.E:

$$\text{Derive an expression } \pi = \frac{1}{2} Ku^2 - Pu \text{ same as previous problem.} \quad \text{---(a)}$$

(b) To determine extreme value of P.E:

$$\text{For extreme value of } \pi, \frac{\partial \pi}{\partial u} = 0$$

Therefore, differentiate Eqn. (a) w.r.t u , we get

$$Ku - P = 0$$

$$\therefore u = \frac{P}{K} \quad \text{---(b)}$$

$$\text{where } K = \frac{AE}{l} = \frac{t * b * E}{l} = \frac{10 * 20 * 200 * 10^3}{100} = 4 * 10^5 \text{ N/mm} \quad \therefore A = t * b$$

$$[E = 200 \text{ Gpa} = 200 \times 10^3 \text{ Mpa or N/mm}^2]$$

Substitute the values of P and K into Eqn. (b), we get the extreme value of u .

$$u = \frac{P}{K} = \frac{3 \times 10^3}{4 \times 10^5} = 7.5 \times 10^{-3} \text{ mm}$$

Substitute for the above equation, we get the extreme value of P.E as

$$\pi = \frac{1}{2} 4 \times 10^5 * (7.5 \times 10^{-3})^2 - 3 \times 10^3 * 7.5 \times 10^{-3}$$

$$\pi = -11.25 \text{ N-mm}$$

(c) Equilibrium stability:

If internal force Ku is equal to external force, the bar is under equilibrium.

$$\text{Here, } Ku = 4 \times 10^5 * 7.5 \times 10^{-3} = 3 \times 10^3 \text{ N} = P$$

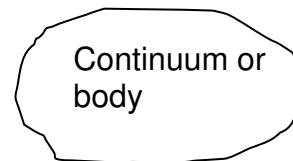
Therefore system is under equilibrium.

8. Continuum Methods

The best way to solve any simple physical problems governed by the differential equations is to obtain the analytical solutions. There are many situations for complex problems, the analytical solution is difficult to obtain. In such cases, some numerical or approximate methods known as continuum methods are used to obtain an approximate solution. The commonly used continuum methods are:

1. Rayleigh-Ritz method
2. Galerkin method and
3. Variational method

In continuum methods, the solution of a domain or body or structure is obtained by subdividing the domain into infinitesimal number of points. The differential equations governing the behavior of a typical infinitesimal domain are given along with boundary condition and an approximate solution field is defined over the entire domain. However, the available mathematical techniques used in these methods usually limit the possibility of simplification of the problems those having complex material property and boundary conditions.



Difference between continuum methods and FEM:

SI No	Continuum Methods	FEM
1	The domain is subdivided into infinitesimal number of points.	The domain is subdivided into finite number of points.
2	An approximate solution field is defined over the entire domain	An approximate solution field is defined in piecewise fashion over subdomains called finite elements.
3	The differential equations governing the behavior of the domain are given along with the boundary conditions over entire domain.	These equations are given over a finite element and the overall system equations are obtained through assembly. The boundary conditions are applied for this overall equation.
4	The available mathematical techniques used in these methods usually limit the possibility of simplification of the problems those having complex material property and boundary conditions.	FEM simplifying the problems of complex material property and boundary conditions.
5	Solutions to complex geometry	Complex geometries are subdivided

	cannot be obtained, only for simple shaped geometries the solutions can be obtained.	into finite number of simple shaped geometries and hence solutions can be obtained.
6	Principle of minimum potential energy or the principle of virtual work are applied to whole domain to ensure stable equilibrium of the overall domain.	These principles are applied to each of the elements to ensure stable equilibrium of each elements.
7	Solutions are obtained for whole domain	Solutions obtained are element wise. The elemental solutions are then assembled to get overall solutions.
8	It is difficult to handle both material and geometric non-linearities.	Techniques are available to convert non-linearities into linearity.

Rayleigh-Ritz method:

It is an approximate numerical technique in which the total potential energy π is used to obtain approximate solution of the continuum. This method involves the construction of an assumed displacement field, say

$$\begin{aligned}
 u &= \sum a_i \phi_i(x, y, z) & i &= 1 \text{ to } l \\
 v &= \sum a_j \phi_j(x, y, z) & j &= l+1 \text{ to } m \\
 w &= \sum a_k \phi_k(x, y, z) & k &= m+1 \text{ to } n
 \end{aligned}
 \quad \text{---(a)}$$

The functions ϕ_i, ϕ_j, ϕ_k are usually taken as polynomials. a_i, a_j, a_k are constants which are to be determined. The displacement fields u, v, w must satisfy specified boundary conditions.

The total potential energy of a 3D structure is expressed as

$$\pi = \frac{1}{2} \iiint_V \{\sigma\}^T \{\epsilon\} dV - \iiint_V \{u\}^T \{f\} dV - \iint_S \{u\}^T \{T\} dS - \sum \{u_i\}^T \{P_i\} \quad \text{---(b)}$$

Now introducing stress-strain and strain-displacement relations and substitution of Eqn. (a) into above Eqn. (b) yields the total potential energy as a functions of constants a 's.

i.e. $\pi = \pi(a_1, a_1, \dots, a_r)$

Then, the extremum of π w.r.t a_i ($i = 1$ to r) yields the set of r equilibrium equations. $\frac{\partial \pi}{\partial a_i} = 0$

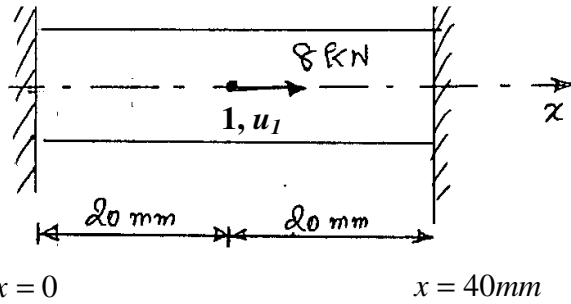
$$(i = 1, 2, \dots, r)$$

The following steps are used to obtain the solution of a problem in Rayleigh-Ritze method:

1. Formulate potential energy functional for a given continuum (body).
2. Assume a trial displacement function for the given continuum. The function assumed should satisfy the boundary conditions.
3. Substitute the displacement function into potential energy functional.
4. Minimize the potential energy functional so as to get the equilibrium conditions.
5. Determine the unknown displacements and hence the stresses and strains.

Prob. 1: Use Rayleigh-Ritz method to find the displacement and stress at the mid point of the rod shown in Fig. The area of cross section of the bar is 400 mm^2 and the modulus of elasticity of the material is 70 Gpa . Assume the displacement to be second degree polynomial.

(VTU Feb. 2002)



Solution: Let point 1 be the point of application of load or force (here mid-point of the bar) and u_1 be the corresponding displacement. Then, the displacement and stress at mid point of bar can be obtained using the following steps.

Steps:

1. *Formulation of potential energy functional:* The total potential energy of the bar is given by $\pi = S.E + WP$

The strain energy of the bar is expressed as (see note)

$$S.E = \frac{AE}{2} \int_0^l \left(\frac{\partial u}{\partial x} \right)^2 dx = \frac{400 \times 70 \times 10^3}{2} \int_0^{40} \left(\frac{\partial u}{\partial x} \right)^2 dx = 14 \times 10^6 \int_0^{40} \left(\frac{\partial u}{\partial x} \right)^2 dx \quad N-mm.$$

Since the bar is subjected to only point load, the work potential is expressed as,

$$WP = -P_1 u_1 = -8 \times 10^3 u_1 \quad N-mm$$

Substituting into π , we get

$$\pi = 14 \times 10^6 \int_0^{40} \left(\frac{\partial u}{\partial x} \right)^2 dx - 8 \times 10^3 u_1 \quad \text{---(a)}$$

2. *Assumption of trial displacement function:* Assume a 2nd degree displacement function

$$u = a_1 + a_2 x + a_3 x^2 \quad \text{---(b)}$$

The above function should satisfy the following boundary conditions.

At $x=0$, $u=0$ and at $x=40mm$, $u=0$. Substitute 1st B.C into Eqn.(b), we get

$$0 = a_1 + 0 + 0 \quad \therefore a_1 = 0$$

Similarly, substitution of 2nd B.C into Eqn. (b) yields,

$$0 = 0 + a_2 \times 40 + a_3 \times 40^2 \quad \text{or } a_2 = -40a_3$$

Substitute a_1 and a_2 into Eqn. (b), we get

$$u = -40a_3 x + a_3 x^2 = a_3 (x^2 - 40x) \quad \text{---(c)}$$

3. *Substitution of the displacement function into potential energy functional:*

$$\text{we have, } \pi = 14 \times 10^6 \int_0^{40} \left(\frac{\partial u}{\partial x} \right)^2 dx - 8 \times 10^3 u_1$$

Differentiate Eqn. (c) w.r.t x , we get $\frac{\partial u}{\partial x} = a_3 (2x - 40)$.

Substitute $x=20$ mm into Eqn. (c), we get the displacement at midpoint as

$$u_1 = a_3 (20^2 - 40 \times 20) = -400a_3. \text{ Substitute into } \pi, \text{ we get}$$

$$\pi = 14 \times 10^6 \int_0^{40} a_3^2 (2x - 40)^2 dx - 8 \times 10^3 (-400a_3) = 14 \times 10^6 a_3^2 \int_0^{40} (4x^2 + 1600 - 80x) dx + 32 \times 10^5 a_3$$

$$\pi = 14 \times 10^6 a_3^2 \left(4 \frac{x^3}{3} + 40^2 x - 4 \times 40 \frac{x^2}{2} \right)_0^{40} + 32 \times 10^5 a_3$$

$$\pi = 14 \times 10^6 a_3^2 \left(4 * \frac{40^3}{3} + 40^2 * 40 - 4 * 40 * \frac{40^2}{2} \right) + 32 \times 10^5 a_3$$

$$\pi = 298666.7 \times 10^6 a_3^2 + 32 \times 10^5 a_3 \quad \text{---(d)}$$

4. *Minimize the potential energy functional:* Since the potential energy functional (Eqn. (d)) is a function of a_3 (i.e. $\pi = \pi(a_3)$) only, the minimization of potential energy yields

$\frac{\partial \pi}{\partial a_3} = 0$. Therefore, differentiate Eqn. (d) w.r.t a_3 we get

$$2 * 298666.7 \times 10^6 a_3 + 32 \times 10^5 = 0 \quad \text{Or } a_3 = -\frac{32 * 10^5}{2 * 298666.7 * 10^6} = -5.36 * 10^{-6}$$

5. *Determine unknown displacement and stress:*

(a) *Displacement:* Substitute into Eqn. (c), we get an expression for the displacement along the bar

$$u = -5.36 * 10^{-6} (x^2 - 40x) \quad \text{---(e)}$$

Substitute $x = 20\text{mm}$ into the above equation we get the displacement at mid point.

Therefore, displacement at midpoint,

$$u_p = u_{x=20\text{mm}} = -5.36 * 10^{-6} (20^2 - 40 * 20) = 2.144 * 10^{-3} \text{ mm}$$

(b) *Stress:* Stress is given by $\sigma_x = E \varepsilon_x = E \frac{\partial u}{\partial x}$

Differentiate Eqn. (e) and substitute we get an expression for stress along the bar

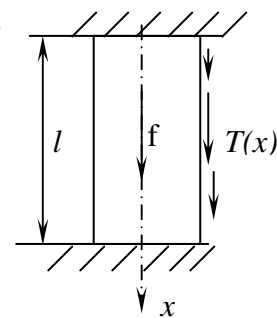
$$\sigma = E(-5.36 * 10^{-6} (2x - 40)) = -70 * 10^3 * 5.36 * 10^{-6} (2x - 40)$$

Then, stress at $x = 0$ is $\sigma_{x=0} = -70 * 10^3 * 5.36 (2 * 0 - 40) = 15 \text{ N/mm}^2$ (tensile)

$$\text{stress mid point is } \sigma_{x=20\text{mm}} = -70 * 5.36 * 10^{-3} (2 * 20 - 40) = 0 \text{ N/mm}^2$$

$$\text{stress at } x = 2 \text{ is } \sigma_{x=40\text{mm}} = -70 * 10^{-3} * 5.36 (2 * 40 - 40) = -15 \text{ N/mm}^2 \text{ (compressive)}$$

Prob. 2: A steel rod is attached to a rigid wall at each ends and is subjected to a distributed load $T(x)$ and a body force f (force/unit volume) as shown in Fig. Determine the displacement at midpoint of the rod using Rayleigh-Ritz method. Assume a second degree polynomial to the displacement field.



$E = 1 \text{ unit}$
 $T(x) = 1 \text{ unit}$
 $f = 1 \text{ unit}$
 $l = 2 \text{ units}$
 $A = 1 \text{ unit}$

Solution: Rod is subjected to body force and surface force only.

Surface force = distributed load $T_x = T(x) = 1 \text{ Force/unit length}$

Body force = $f_x = f = 1 \text{ Force/unit volume}$

1. *Formulation of potential energy functional:* The total potential energy of the bar is given by

$$\pi = S.E + WP$$

The strain energy of the bar is expressed as

$$S.E = \frac{AE}{2} \int_0^l \left(\frac{\partial u}{\partial x} \right)^2 dx = \frac{1 * 1}{2} \int_0^2 \left(\frac{\partial u}{\partial x} \right)^2 dx \quad \text{N-mm} \quad \because A = E = 1; l = 2$$

Since the bar is subjected to body force and surface force only, the work potential is expressed as,

$$WP = -(WP|_{Body\ Force} + WP|_{Surface\ Force})$$

$$WP = -\int_0^l f_x u A dx - \int_0^l T_x u dx = -\int_0^2 1 * 1 * u dx - \int_0^2 1 * u dx = -2 \int_0^2 u dx$$

$$\because f_x = 1; T_x = T(x) = 1; A = 1$$

Substituting into π , we get

$$\pi = \frac{1}{2} \int_0^2 \left(\frac{\partial u}{\partial x} \right)^2 dx - 2 \int_0^2 u dx \quad \text{---(a)}$$

Note: In general work potential due to body and surface forces are expressed as

$$WP|_{body\ Force} = -\int_V u^T f dV \quad \text{and} \quad WP|_{Surface\ Force} = -\int_S u^T T dS$$

where $u^T = \{u \quad v \quad w\}$; $f = \{f_x \quad f_y \quad f_z\}^T$; $T = \{T_x \quad T_y \quad T_z\}^T$ For 3D problem

$u^T = \{u \quad v\}$; $f = \{f_x \quad f_y\}^T$; $T = \{T_x \quad T_y\}^T$ For 2D problem

$u^T = \{u\}$; $f = \{f_x\}^T$; $T = \{T_x\}^T$ For 1D problem

and $dV = A dx$ for 1D bar and $dS = dx$ if surface force is UDL or force/unit length. Therefore, we get the WP due to body and surface force components for 1D bar as

$$WP|_{body\ Force} = -\int_0^l f_x u A dx \quad \text{and} \quad WP|_{Surface\ Force} = -\int_0^l T_x u dx$$

2. Assumption of trial displacement function: Assume a 2nd degree displacement function

$$u = a_1 + a_2 x + a_3 x^2 \quad \text{---(b)}$$

The above function should satisfy the following boundary conditions.

At $x=0$, $u=0$ and at $x=2$, $u=0$. Substitute 1st B.C into Eqn.(b), we get

$$0 = a_1 + 0 + 0 \quad \therefore a_1 = 0$$

Similarly, substitution of 2nd B.C into Eqn. (b) yields,

$$0 = 0 + 2a_2 + 2^2 a_3 \quad \text{or} \quad a_2 = -2a_3$$

Substitute a_1 and a_2 into Eqn. (b), we get

Substitute a_1 and a_2 into Eqn. (b), we get

$$u = a_3(x^2 - 2x) \quad \text{---(c)}$$

3. Substitution of the displacement function into potential energy functional:

Differentiate Eqn. (c) w.r.t x we get

$$\frac{\partial u}{\partial x} = a_3(2x - 2) = 2a_3(x - 1). \text{ Substitute } \frac{\partial u}{\partial x} \text{ and } u \text{ into Eqn. (a), we get}$$

$$\pi = \frac{1}{2} \int_0^2 2a_3(x-1)^2 dx - 2 \int_0^2 a_3(x^2 - 2x) dx = \frac{1}{2} \int_0^2 4a_3^2(x^2 + 1 - 2x) dx - 2 \int_0^2 a_3(x^2 - 2x) dx$$

$$\pi = 2a_3^2 \int_0^2 (x^2 + 1 - 2x) dx - 2a_3 \int_0^2 (x^2 - 2x) dx = 2a_3^2 \left(\frac{x^3}{3} + x - 2\frac{x^2}{2} \right)_0^2 - 2a_3 \left(\frac{x^3}{3} - 2\frac{x^2}{2} \right)_0^2$$

$$\pi = 2a_3^2 \left(\frac{2^3}{3} + 2 - 2^2 \right) - 2a_3 \left(\frac{2^3}{3} - 2^2 \right) = 1.333a_3^2 + 2.667a_3 \quad \text{---(d)}$$

4. *Minimize the potential energy functional:* Since the potential energy functional (Eqn. (d)) is a function of a_3 (i.e. $\pi = \pi(a_3)$) only, the minimization of potential energy yields $\frac{\partial \pi}{\partial a_3} = 0$.

$$\therefore 2 * 1.333a_3 + 2.667 = 0 \quad \text{Or } a_3 = -\frac{2.667}{2 * 1.333} = -1$$

5. *Determine unknown displacement and stress:*

(a) *Displacement:* Substitute a_3 into Eqn. (c), we get an expression for the displacement along the bar

$$u = -1 * (x^2 - 2x) = 2x - x^2 \quad \text{---(e)}$$

Substitute $x = 1 \text{ unit}$ into the above equation we get the displacement at mid point.

Therefore, displacement at midpoint, $u_p = 2 * 1 - 1 = 1 \text{ unit}$

$$(b) \text{ Stress: Stress is given by } \sigma_x = E\varepsilon_x = E \frac{\partial u}{\partial x}$$

Differentiate Eqn. (e) and substitute we get an expression for stress along the bar

$$\sigma = E(2 - 2x)$$

Then, stress at $x = 0$ is $\sigma_{x=0} = 2 - 2 * 0 = 2 \text{ unit}$

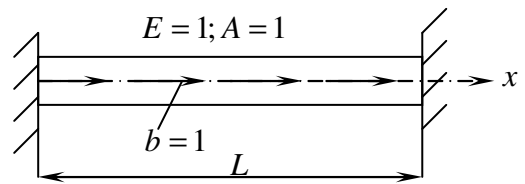
stress mid point is $\sigma_{x=1} = 2 - 2 * 1 = 0 \text{ unit}$

stress at $x = 2$ is $\sigma_{x=2} = 2 - 2 * 2 = -2 \text{ unit}$

Prob. 3: Use Rayleigh-Ritz method to find the displacement of the midpoint of the rod shown in Fig. Body force per unit volume = 1. Use second degree polynomial approximation for displacement. (VTU Jan 08)

Solution: Here, the rod is subjected to body force only.

$$\therefore WP = -A \int_0^L f_x u dx = -A \int_0^L b u dx = - \int_0^L u dx \quad \because A = b = 1$$



$$\pi = \frac{1}{2} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx - \int_0^L u dx \quad \text{---(a)}$$

$$u = a_1 + a_2 x + a_3 x^2 \quad \text{---(b)}$$

Boundary conditions are, at $x = 0$, $u = 0$ and at $x = L$, $u = 0$

$$\text{Hence, we get } a_1 = 0 \text{ and } a_2 = -La_3 \quad \therefore u = a_3(x^2 - Lx) \quad \text{---(c)}$$

Differentiate Eqn. (c) w.r.t x and substitute into Eqn. (a) we get

$$\pi = \frac{1}{2} a_3^2 \int_0^L (4x^2 + L^2 - 4Lx) dx - \int_0^L a_3(x^2 - Lx) dx = \frac{1}{2} a_3^2 \left(\frac{4L^3}{3} + L^3 - \frac{4L^3}{2} \right) - a_3 \left(\frac{L^3}{3} - \frac{L^3}{2} \right)$$

$$\pi = \frac{L^3}{6} a_3^2 + \frac{L^3}{6} a_3 \quad \text{---(d). Minimizing Eqn. (d), we get}$$

$$\frac{\partial \pi}{\partial a_3} = \frac{L^3 2a_3}{6} + \frac{L^3}{6} = 0 \quad \text{Or } a_3 = -0.5. \text{ Substitute into Eqn. (c), we get}$$

$$u = 0.5(Lx - x^2) \quad \text{Then, } U_{x=L/2} = 0.5 \left(L * \frac{L}{2} - \frac{L^2}{4} \right) = 0.125L^2$$

Prob. 4: A bar of length 'L' is subjected to a body force as shown in Fig. If the density of material of the bar is 0.1 kg/m^3 , find the displacements, stresses and strains at the midpoint and free end of the bar.

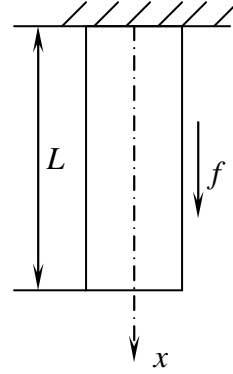
Solution:

$$\text{Body force, } f_x = f = \rho g = 0.1 * 9.81 = 0.981 \text{ N/m}^3$$

$$E = 200 * 10^9 \text{ Pa or N/m}^2$$

Steps:

$$1. \pi = \frac{AE}{2} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx - A \int_0^L (\rho g) u dx$$



$$A = 100 \text{ m}^2 \\ E = 200 \text{ GPa}$$

$$\pi = \frac{100 * 200 * 10^9}{2} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx - 100 \int_0^L 0.981 * u dx = 10^{13} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx - 100 \int_0^L 0.981 * u dx$$

---(a)

$$2. \text{ Assume } u = a_1 + a_2 x + a_3 x^2$$

---(b)

$$\text{Boundary condition, at } x = 0, u = 0 \quad \therefore a_1 = 0 \quad \therefore u = a_2 x + a_3 x^2$$

---(c)

$$3. \text{ Differentiate Eqn. (c) w.r.t } x, \text{ we get } \frac{\partial u}{\partial x} = a_2 + 2a_3 x. \text{ Substitute into Eqn. (a), we get}$$

$$\pi = 10^{13} \int_0^L (a_2 + 2a_3 x)^2 dx - 98.1 \int_0^L (a_2 x + a_3 x^2) dx$$

$$\pi = 10^{13} \int_0^L (a_2^2 + 4a_2 a_3 x + 4a_3^2 x^2) dx - 98.1 \int_0^L (a_2 x + a_3 x^2) dx$$

$$\pi = 10^{13} \left(a_2^2 L + 4a_2 a_3 \frac{L^2}{2} + 4a_3^2 \frac{L^3}{3} \right) - 98.1 \left(a_2 \frac{L^2}{2} + a_3 \frac{L^3}{3} \right) \quad \text{---(d)}$$

$$4. \text{ Since } \pi = \pi(a_2, a_3), \text{ minimization of Eqn. (d) yields}$$

$$\frac{\partial \pi}{\partial a_2} = 10^{13} \left(2a_2 L + 4a_3 \frac{L^2}{2} \right) - 98.1 \left(\frac{L^2}{2} \right) = 0 \quad \text{Or } 2a_2 + 2a_3 L = \frac{98.1L}{10^{13}} \quad \text{---(i)}$$

$$\frac{\partial \pi}{\partial a_3} = 10^{13} \left(8a_3 \frac{L^3}{3} + 4a_2 \frac{L^2}{2} \right) - 98.1 \left(\frac{L^3}{3} \right) = 0 \quad \text{Or } 2a_2 + \frac{8}{3} a_3 L = \frac{98.1L}{3 * 10^{13}} \quad \text{---(ii)}$$

Solve Eqns. (i) and (ii), i.e. subtract Eqn. (i) from (ii), we get

$$\left(\frac{8}{3} - 2\right)a_3L = \left(\frac{98.1}{3} - 49.05\right)\frac{L}{10^{13}} \quad \text{Or } a_3 = -24.525 * 10^{-13} m. \text{ Substitute into Eqn. (i), we get}$$

$$a_2 = 49.05 * 10^{-13} L$$

5. Substitute a_2 and a_3 into Eqn. (c), we get $u = 49.05 * 10^{-13} (Lx - 0.5x^2) m$ ---(e)

Displacements:

$$\text{Displacement at midpoint (at } x=L/2), u_{x=L/2} = 49.05 * 10^{-13} \left(\frac{L^2}{2} - 0.5 \frac{L^2}{4} \right) = 18.394 * 10^{-13} L^2$$

$$\text{Displacement at free end (i.e. at } x=L), u_{x=L} = 49.05 * 10^{-13} (L^2 - 0.5L^2) = 24.525 * 10^{-13} L^2$$

$$\text{Strains: we know that, strain } \varepsilon = \frac{\partial u}{\partial x} = 49.05 * 10^{-13} (L - 2 * 0.5x)$$

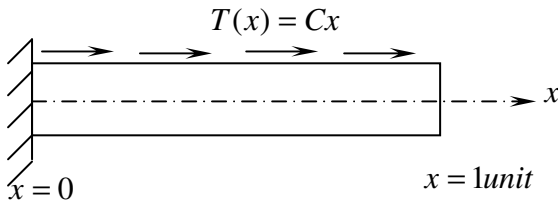
$$\text{Strain at midpoint, } \varepsilon_{x=L/2} = 49.05 * 10^{-13} \left(L - \frac{L}{2} \right) = 49.05 * 10^{-13} * \frac{L}{2}$$

$$\text{Strain at free end, } \varepsilon_{x=L} = 49.05 * 10^{-13} (L - L) = 0$$

Stresses: We know that, stress $\sigma = E\varepsilon$

$$\text{Stresses at midpoint and free end are, } \sigma_{x=L/2} = 200 * 10^9 * 49.05 * 10^{-13} * \frac{L}{2} \text{ and } \sigma_{x=L} = 0$$

Prob. 5: For the system shown in Fig., the distributed load varies linearly, determine the displacements and stresses at midpoint and free end of the rod. Take, $A=E=C=1$ unit.



Hints: The rod is subjected surface force only and $T(x) = Cx = x \quad \therefore C = 1$

$$\therefore \pi = \frac{1}{2} \int_0^1 \left(\frac{\partial u}{\partial x} \right)^2 dx - \int_0^1 x u dx. \text{ Proceed similar to the previous problem to show, } u = a_2 x + a_3 x^2$$

and

$$\pi = \frac{1}{2} \int_0^1 (a_2 + 2a_3 x)^2 dx - \int_0^1 x(a_2 x + a_3 x^2) dx = \frac{1}{2} \int_0^1 (a_2^2 + 4a_3^2 x^2 + 4a_2 a_3 x) dx - \int_0^1 (a_2 x^2 + a_3 x^3) dx$$

Proceed to determine a_2 and a_3